THE NILPOTENCE CLASS OF THE FRATTINI SUBGROUP

BY

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ABSTRACT

A p-group of sufficiently large nilpotence class cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group. The Frattini subgroup of a group of order $\Pi_{p_i}^{\alpha_i}$ with max α_i at least 3, has nilpotence class at most $\frac{1}{2}(\max \alpha_i - 1)$. The Frattini subgroup of a t-group is abelian. The occurrence of groups of order p^4 as normal subgroups contained in Frattini subgroups is investigated.

Only finite groups are considered. The ascending central series of a group G is denoted by $1 = Z_0(G) \leqq Z_1(G) \leqq \cdots$. If $Z_i(G) = G$ for some i, G is called nilpotent and the smallest such i is the nilpotence class of G, denoted by $\mathrm{cl}(G)$. If $|G|$, the order of G, is p^n with p a prime, then G is nilpotent and (provided n is at least 2) $1 \leqq \mathrm{cl}(G) \leqq n-1$. If $\mathrm{cl}(G) = n - 1 \geqq 2$, G is said to have maximal class. The Frattini subgroup of G, denoted by $\Phi(G)$, is the intersection of the maximal proper subgroups of G and is a nilpotent characteristic subgroup. When a group is described by generators and relations, only those commutators which are different from 1 will be explicitly given.

1. The class of $\Phi(G)$.

Let Γ be a class of groups, closed under the formation of finite direct products. If H is nilpotent, then there exists $G \in \Gamma$ with $H \lhd G$ and $H \leqq \Phi(G)$ if and only if each Sylow subgroup of H has that same property. Hence, in considering the question of which nilpotent groups occur as normal subgroups contained in

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the Frattini subgroup of some Γ-group, it is sufficient to consider only the case that H is a p-group.

With regard to this question when Γ is the class of all finite groups, a p-group of nilpotence class at least 3 and with center of order p cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group [5, 7, 8]. Since a p-group of maximal class necessarily has center of order p, such a group cannot occur in the above manner, provided it is of order at least p^4.

We show first that no nonabelian group of order p^3 occurs as a normal subgroup contained in the Frattini subgroup of a finite group, and hence that no p-group of maximal class occurs. We then show that p-groups of sufficiently large nilpotence class cannot occur and obtain an upper bound for $\mathrm{cl}(\Phi(G))$ for arbitrary G, the bound depending only on the order of G.

LEMMA 1. *A nonabelian group of order p^3 cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group.*

PROOF. Let H be nonabelian of order p^3 and let $A(H)$ and $I(H)$ denote respectively the automorphism group and the group of inner automorphisms of H. If $p = 2$, then $I(H) \nleqq \Phi(A(H))$ and the result follows [6, Satz III. 3.13]. If $H = \langle x, y \mid x^{p^2} = y^p = 1, [x, y] = x^p \rangle$, p odd, then $\langle x^p, y \rangle$ is a noncentral characteristic subgroup of order p^2 and the result follows again in this case [5]. Finally, suppose p is odd and H is the group $\langle x, y, z \mid x^p = y^p = z^p = 1, [x, y] = z \rangle$. Each element is uniquely representable as $x^a y^b z^c$ with $0 \leqq a, b, c \leqq p-1$. Let $S = \{xy^b \mid 0 \leqq b \leqq p-1\}$ and $M = \{\sigma \in A(H) \mid \sigma(S) = S\}$. M is a subgroup of $A(H)$ and $M \cap I(H)$ is trivial, i.e. $MI(H)$ splits over $I(H)$. By two results of Gaschütz [6, Hauptsatz I.17.4, Satz III. 3.13], it suffices to show that $[A(H): MI(H)]$ is p-free, for then $A(H)$ splits over $I(H)$. There is an automorphism σ of H such that $\sigma(x) = xy$ and $\sigma(y) = y$. Clearly $\sigma \in M$ and $|\sigma| = p$. Hence $|M|$ is a multiple of p. Because the p-share of $|A(H)|$ is p^3 and $|I(H)| = p^2$, the result follows.

COROLLARY. *A p-group of maximal class cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group.*

LEMMA 2. *If H is a nonabelian p-group such that $[Z_i(H): Z_{i-1}(H)] = p$ for some $i \neq \mathrm{cl}(H) - 1$, then H cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group.*

PROOF. Consider $H/Z_{i-1}(H)$. If $H \lhd G$ and $H \leqq \Phi(G)$, then $Z_{i-1}(H) \lhd G$,

$H/Z_{i-1}(H) \lhd G/Z_{i-1}(H)$ and $H/Z_{i-1}(H) \leqq \Phi(G/Z_{i-1}(H))$. Thus $\mathrm{cl}(H/Z_{i-1}(H)) \leqq 2$ and $i = \mathrm{cl}(H)$. But then $[H: Z_{cl(H)-1}(H)] = p$, which is impossible.

We now consider p-groups of nilpotence class which is "large" in the following sense.

DEFINITION. Let $|H| = p^n$. H is of large class if $n \geqq 2$ and $\mathrm{cl}(H) > \frac{1}{2}n$.

THEOREM 1. *A p-group of large class cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group.*

PROOF. Let $|H| = p^n$, $H \lhd G$ and $H \leqq \Phi(G)$. If $\mathrm{cl}(H) \leqq 2$, then either $\mathrm{cl}(H)$ is not large or $n = 3$. If $n = 3$, Lemma 1 implies the desired result. Let $\mathrm{cl}(H) = k \geqq 3$. Then $p^n = |H| = \Pi[Z_i(H): Z_{i-1}(H)] \geqq p^{2k-1}$, for Lemma 2 implies that $[Z_i(H): Z_{i-1}(H)] \geqq p^2$ except possibly when $i = k-1$. Thus $k \leqq \frac{1}{2}(n+1)$. If $k = \frac{1}{2}(n+1)$, then $H/Z_{k-2}(H)$ is nonabelian and of order p^3. But $H/Z_{k-2}(H)$ is normal in, and contained in the Frattini subgroup of, $G/Z_{k-2}(H)$. This contradicts Lemma 1, and so $k \leqq \frac{1}{2}n$.

COROLLARY. *Let G be a p-group with derived group G' of order p^n, $n \geqq 2$. Then* $\mathrm{cl}(G') \leqq \frac{1}{2}n$.

The above corollary is a known result [4, Section 2] and follows easily by commutator arguments.

Theorem 1 leads to the following upper bound for $\mathrm{cl}(\Phi(G))$.

THEOREM 2. *If $|G| = \Pi p_i^{\alpha_i}$, where the p_i are distinct primes and* $\max \alpha_i$ *is at least* 3, *then* $\mathrm{cl}(\Phi(G)) \leqq \frac{1}{2}(\max \alpha_i - 1)$.

PROOF. Let $|\Phi(G)| = \Pi p_i^{n_i}$ and let P_i be the Sylow p_i-subgroup of $\Phi(G)$. Then $P_i \lhd G$ and so $\mathrm{cl}(P_i) \leqq \frac{1}{2}n_i$, unless $n_i = 0, 1$. But $\mathrm{cl}(\Phi(G)) = \max \mathrm{cl}(P_i)$ and [6, Satz III. 3.8] $n_i \leqq \alpha_i - 1$. Hence, $\max \mathrm{cl}(P_i) \leqq \frac{1}{2}(\max \alpha_i - 1)$.

Note that if $\max \alpha_i$ is less than 3, $\Phi(G)$ is necessarily abelian.

2. Frattini subgroups of t-groups

Let G be a finite group, all of whose Sylow subgroups are abelian. Then the Sylow subgroups of $\Phi(G)$ are abelian and, since $\Phi(G)$ is nilpotent, $\Phi(G)$ is abelian. Consider the following two subclasses of the class of groups having abelian Sylow subgroups: abelian groups, and groups with cyclic Sylow subgroups.

A group G is called a t-group if every subnormal subgroup of G is, in fact, normal in G. Clearly, abelian groups are t-groups. It is also true [2, 3] that groups with cyclic Sylow subgroups have all of their normal subgroups characteristic, and hence are t-groups. In this section, we show that the property of possessing an abelian Frattini subgroup is shared by all of the t-groups.

A group is called Dedekind if each of its subgroups is normal, and a non-abelian Dedekind group, e.g. the quaternion group Q of order 8, is called a Hamiltonian group. A group is Hamiltonian if and only if [6, Satz III. 7.12] it is the direct product of a quaternion group Q, an abelian group of odd order and an elementary abelian 2-group. Since every subgroup of a nilpotent group is subnormal, each nilpotent t-group is a Dedekind group. Finally, we remark that as a consequence of Lemma 1, the quaternion group Q cannot occur as a normal subgroup contained in the Frattini subgroup of any finite group.

THEOREM 3. *The Frattini subgroup of a t-group is abelian.*

PROOF. Let G be a t-group. Since the t-group property is inherited by normal subgroups, $\Phi(G)$ is a nilpotent t-group and hence Dedekind. If $\Phi(G)$ were Hamiltonian, it would have Q as a direct factor and, since G is a t-group, Q would be a normal subgroup of G. This contradicts Lemma 1 and thus $\Phi(G)$ is abelian.

3. The occurrence of groups of order p^4 as normal subgroups contained in Frattini subgroups

Let p be an odd prime and let H be a nonabelian group of order p^4. Bechtell [1] has determined which such groups occur as Frattini subgroups of p-groups. We are able to give a partial answer to the more general question of which such groups occur as normal subgroups contained in Frattini subgroups of arbitrary finite groups.

The following two groups occur as Frattini subgroups of p-groups [1]: $\langle x,y,z \mid x^{p^2}=y^p=z^p=1,\ [x,z]=x^p\rangle$, and $\langle x,y,z,w \mid x^p=y^p=z^p=w^p=1,$ $[z,w]=x\rangle$. The group $\langle x,y \mid x^{p^3}=y^p=1,\ [x,y]=x^{p^2}\rangle$ possesses a non-central characteristic subgroup $\langle x^{p^2},y\rangle$ of order p^2 and so [5] does not occur normally and contained in a Frattini subgroup. The group $\langle x,y,z \mid x^{p^2}=y^p$ $=z^p=1,\ [y,z]=x^p\rangle$ possesses a nonabelian characteristic subgroup $\langle x^p,y,z\rangle$ of order p^3 and the group $\langle x,y,z \mid x^{p^2}=y^p=z^p=1,\ [x,z]=y\rangle$ possesses a characteristic subgroup $\langle x^p\rangle$ whose factor group is nonabelian and of order p^3. Hence, neither of these groups can occur. Four of the remaining five nonabelian groups of order p^4 are of maximal class and consequently do not occur. The one remaining group, $\langle x,y \mid x^{p^2}=y^{p^2}=1,\ [x,y]=x^p\rangle$ has cyclic center and so [1] cannot occur as a normal subgroup contained in the Frattini subgroup of any p-group. Now, if P is a normal Sylow p-subgroup of a group G, then $\Phi(P)$ is the Sylow p-subgroup of $\Phi(G)$. Hence this group cannot occur as a normal subgroup contained in the Frattini subgroup of any group which is p-closed (i.e. which has normal Sylow p-subgroup). For this group, the general question remains unanswered.

For groups of order 2^4 we have the following. The three groups of maximal class do not occur. Each of the groups $\langle x, y, z \mid x^4 = y^2 = z^2 = 1, [x, z] = x^2 \rangle$, $\langle x, y, z \mid x^4 = y^4 = z^2 = 1, [x, y] = x^2 = y^2 \rangle$ occurs as the Frattini subgroup of a group of order 2^6. The group $\langle x, y \mid x^8 = y^2 = 1, [x, y] = x^4 \rangle$ possesses a non-central characteristic subgroup $\langle x^2, y \rangle$ of order 2^2 and so [5] does not occur. The groups $\langle x, y, z \mid x^4 = y^2 = z^2 = 1, [y, z] = x^2 \rangle$, and $\langle x, y \mid x^4 = y^4 = 1, [x, y] = x^2 \rangle$, having cyclic centers, cannot occur in any 2-closed group, but the general question is unanswered for these. For the one remaining nonabelian group of order 2^4: $\langle x, y, z \mid x^4 = y^2 = z^2 = 1, [x, z] = y \rangle$, we have no information at all.

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